

Fisher Information of Scale

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Abstract

Motivated by the information bound for the asymptotic variance of M-estimates for scale, we define Fisher information of scale of any distribution function F on the real line as the supremum of all $(\int x \phi'(x) F(dx))^2 / \int \phi^2(x) F(dx)$, where ϕ ranges over the continuously differentiable functions with derivative of compact support and where, by convention, $0/0 := 0$. In addition, we enforce equivariance by a scale factor. Fisher information of scale is weakly lower semicontinuous and convex. It is finite iff the usual assumptions on densities hold, under which Fisher information of scale is classically defined, and then both classical and our notions agree. Fisher information of scale finite is also equivalent to L_2 -differentiability and local asymptotic normality, respectively, of the scale model induced by F .

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1. Motivation and Definition

If F is any distribution function on \mathbb{R} , the real line, and $\phi: \mathbb{R} \rightarrow \mathbb{R}$ a suitable scores function such that $\int \phi dF = 0$, an M-estimate of scale S_n may formally be defined by

$$\sum_{i=1}^n \phi\left(\frac{x_i}{S_n}\right) = 0. \quad (1.1)$$

The estimand refers to the scale model $(F_\sigma)_{0 < \sigma < \infty}$ induced by $F = F_1$, where $F_\sigma(x) = F(x/\sigma)$.

Taylor expanding $\phi(x/s) = \phi(x/\sigma) - (s - \sigma)\phi'(x/\sigma)x/\sigma^2 + \dots$, we formally obtain

$$\sqrt{n}(S_n - \sigma) = \sigma \frac{n^{-1/2} \sum_1^n \phi(x_i/\sigma)}{n^{-1} \sum_1^n \phi'(x_i/\sigma) x_i/\sigma} + \dots \quad (1.2)$$

such that under observations x_1, \dots, x_n i.i.d. $\sim F_\sigma$ and assuming sufficient regularity, in particular consistency, $\sqrt{n}(S_n - \sigma)$ will as $n \rightarrow \infty$ be asymptotically normal with mean zero and variance

$$V(\phi, F_\sigma) = \sigma^2 V_1(\phi, F), \quad V_1(\phi, F) := \frac{\int \phi^2(x) F(dx)}{(\int x \phi'(x) F(dx))^2}. \quad (1.3)$$

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If ϕ is differentiable with continuous derivative of compact support, both $\phi(x)$ and $x\phi'(x)$ are bounded, so the integrals in (1.3) are well-defined for any distribution F on the Borel σ -algebra \mathbb{B} of \mathbb{R} . As in the theory of generalized functions (Rudin (1991, Ch. 6)), regularity conditions are shifted to the test functions whenever possible.

The usual information bound for asymptotic variance would say that $V(\phi, F_\sigma) \geq \mathcal{J}_s^{-1}(F_\sigma)$ and, hopefully, the lower bound will also be achieved.

This leads us to the following definition of $\mathcal{J}_{s1}(F)$. The extension to $\mathcal{J}_s(F_\sigma)$ for the scale transforms F_σ of F matches (1.3).

Definition 1.1. *Fisher information of scale, for any distribution F on the real line, is defined by*

$$\mathcal{J}_{s1}(F) := \sup_{\phi \in \mathcal{C}_{c1}} \frac{(\int x \phi'(x) F(dx))^2}{\int \phi^2(x) F(dx)}, \quad (1.4)$$

where \mathcal{C}_{c1} denotes the set of all differentiable functions $\phi: \mathbb{R} \rightarrow \mathbb{R}$ whose derivative is continuous and of compact support, and $0/0 := 0$ by convention. For the scale transforms F_σ of F we define

$$\mathcal{J}_s(F_\sigma) := \sigma^{-2} \mathcal{J}_{s1}(F), \quad 0 < \sigma < \infty. \quad (1.5)$$

Remark 1.2. Since the map $\phi \mapsto \phi_\sigma$, where $\phi_\sigma(x) := \phi(\sigma x)$ and $\phi'_\sigma(x) = \sigma \phi'(\sigma x)$, defines a one-to-one correspondence on \mathcal{C}_{c1} , we obtain scale invariance of \mathcal{J}_{s1} ,

$$\mathcal{J}_{s1}(F_\sigma) = \mathcal{J}_{s1}(F), \quad 0 < \sigma < \infty. \quad (1.6)$$

So extension (1.5) is needed to obtain scale equivariance. In the scale model, as opposed to location, it matters whether a given distribution F is considered element $F = F_1$ or, for example, element $F = F_5$ (in the scale model generated by F_2). \square

Motivated by the information bound, Definition 1.1 is intrinsically statistical. It does not a priori use the assumption of, and suitable conditions on, densities. These properties rather follow from the definition in case \mathcal{J}_s is finite. Another advantage is that Definition 1.1 implies certain topological properties (convexity and lower continuity) of \mathcal{J}_s .

The definition parallels Huber (1981, Def. 4.1) in the location case,

$$\mathcal{J}_1(F) := \sup_{\phi} \frac{(\int \phi'(x) F(dx))^2}{\int \phi^2(x) F(dx)}, \quad (1.7)$$

where ϕ , subject to $\int \phi^2 dF > 0$, ranges over the (smaller) set \mathcal{C}_c^1 of all continuously differentiable functions which themselves are of compact support. \mathcal{J}_1 is shift invariant.

Huber (1981, p. 79), states vague lower semicontinuity and convexity of \mathcal{J}_1 . By Huber (1981, Thm. 4.2), $\mathcal{J}_1(F)$ is finite iff F is absolutely continuous with an absolutely continuous density f such that $f'/f \in L_2(F)$, in which case $\mathcal{J}_1(F) = \int (f'/f)^2 dF$.

Remark 1.3. The latter result, by arguments of the proof to Theorem 2.2 below, still obtains if definition (1.7) is based on \mathcal{C}_{c1} . Only vague lower semicontinuity of \mathcal{J}_1 would be weakened to weak continuity (which, however, makes no difference in the setup of normed measures). The convention $0/0 := 0$ could replace the side condition $\phi \neq 0$ a.e. F in (1.7) as well.

The non-suitability of \mathcal{C}_c^1 , and suitability of \mathcal{C}_{c1} instead, is the tribute to the scale model, for which the functions $x \mapsto x\phi'(x)$ need to be dense in $L_1(F_0)$ with respect to the punctuated (substochastic) measure F_0 introduced in (2.1) below. \square

Fisher information of scale has been treated by Huber (1964, 1981) not in the previous generality but only under suitable assumptions on densities and, in an auxiliary way, has been reduced to the location case by symmetrization and the log-transform, Huber (1981, Sec. 5.6).

2. Main Results

Proposition 2.1. \mathcal{J}_{s1} is weakly lower semicontinuous and convex.

Zero observations do not contain any information about scale. Removing the mass of any distribution F at zero, we define the punctuated, possibly substochastic measure F_0 by

$$F_0 := F - F(\{0\})1_0, \quad (2.1)$$

where 1_0 denotes Dirac measure at 0. In terms of distribution functions, denoting by $1_{[0,\infty)}$ the indicator function, we have $F_0(x) = F(x) - (F(0) - F(0-))1_{[0,\infty)}(x)$.

Theorem 2.2. For any distribution F on the real line, $\mathcal{J}_{s1}(F)$ is finite iff

- i) F_0 is absolutely continuous with a density f such that
- ii) $x \mapsto xf(x)$ is absolutely continuous, and
- iii) $x \mapsto \Lambda(x) := -[xf(x)]'/f(x) \in L_2(F_0)$,

in which case $\mathcal{J}_{s1}(F) = \int \Lambda^2 dF_0 = \int_{x \neq 0} [1 + xf'(x)/f(x)]^2 F(dx)$.

3. Consequences for the Scale Model

For the scale transforms F_σ of F , $\mathcal{J}_{s1}(F_\sigma) = \mathcal{J}_{s1}(F)$ and $\mathcal{J}_s(F_\sigma) = \sigma^{-2} \mathcal{J}_{s1}(F)$ by (1.6) and (1.5), respectively. In particular, $\mathcal{J}_{s1}(F_\sigma)$ and $\mathcal{J}_s(F_\sigma)$ are finite iff $\mathcal{J}_{s1}(F)$ is finite. Also conditions i) and ii) of Theorem 2.2 are simultaneously fulfilled for a density f of F_0 and the density $f_\sigma(x) = \sigma^{-1}f(x/\sigma)$ of the punctuation $F_{\sigma,0}$ of F_σ . In the finite case, since $[xf_\sigma(x)]'/f_\sigma(x)$ in condition iii) of Theorem 2.2 is just $\Lambda(x/\sigma)$, this theorem yields $\mathcal{J}_{s1}(F_\sigma) = \int \Lambda^2(x/\sigma) F_{\sigma,0}(dx)$, which is $\int \Lambda^2(x) F_0(dx) = \mathcal{J}_{s1}(F)$; that is, (1.6) again. Therefore, in the finite case,

$$\mathcal{J}_s(F_\sigma) = \int \Lambda_\sigma^2 dF_{\sigma,0}, \quad 0 < \sigma < \infty. \quad (3.1)$$

the representation of $\mathcal{J}_s(F_\sigma)$ in terms of the usual score function Λ_σ ,

$$\Lambda_\sigma(x) := \frac{1}{\sigma} \Lambda\left(\frac{x}{\sigma}\right) = \frac{\partial}{\partial \sigma} \log f_\sigma(x) = -\frac{1}{\sigma} \left(1 + \frac{x}{\sigma} \frac{f'(\frac{x}{\sigma})}{f(\frac{x}{\sigma})}\right). \quad (3.2)$$

As an analogue to a lemma due to Hájek (1972) in the location case, Swensen (1980, Ch.2, Sec.3) for an absolutely continuous F has shown that conditions i)–iii) of Theorem 2.2 even imply L_2 -differentiability (Rieder, 1994, Def. 2.3.6) of the scale model,

$$\left\| \sqrt{dF_{\sigma+t}} - \sqrt{dF_\sigma} \left(1 + \frac{1}{2} t \Lambda_\sigma\right) \right\| = o(t) \quad \text{as } t \rightarrow 0 \quad (3.3)$$

at $\sigma = 1$ and, by invariance, at any $0 < \sigma < \infty$. By definition, L_2 -differentiability already entails that $\int \Lambda_\sigma^2 dF_\sigma < \infty$. Setting $\Lambda(0) := 0$, we may extend his result to $F(\{0\}) > 0$.

Proposition 3.1. Assume that $\mathcal{J}_{s1}(F) < \infty$. Then the scale model $(F_\sigma)_{0 < \sigma < \infty}$ is L_2 -differentiable with derivative Λ_σ at every $0 < \sigma < \infty$.

L_2 -differentiability of a parametric model implies an expansion of the log-likelihoods, see e.g. Rieder (1994, Thm. 2.3.5); in our case, for each $h \in \mathbb{R}$,

$$\log dF_{\sigma+h/\sqrt{n}}^n / dF_\sigma^n = \frac{1}{\sqrt{n}} \sum_{i=1}^n h^\tau \Lambda_\sigma(x_i) - \frac{1}{2} h^\tau \mathcal{J}_s(F_\sigma) h + o_{F_\sigma^n}(n^0); \quad (3.4)$$

that is, the scale model is *locally asymptotically normal* (LAN). LAN is the basis of asymptotic optimality results as Hájek's Asymptotic Convolution Theorem and the Local Asymptotic Minimax Theorem, see e.g. Rieder (1994, Thm.'s 3.2.3, 3.3.8) and van der Vaart (1998, Thm.'s 8.8, 8.11). Le Cam (1986, 17.3 Prop. 2) even shows that, in the i.i.d. setup, LAN is equivalent to L_2 -differentiability. Thus we obtain the following result.

Proposition 3.2. The following statements are equivalent:

- i) $\mathcal{J}_s(F_\sigma) < \infty$ at some $0 < \sigma < \infty$.
- ii) The scale model is L_2 -differentiable at some $0 < \sigma < \infty$.
- iii) The scale model has the LAN property (3.4) at some $0 < \sigma < \infty$.

By invariance, the validity of each statement at one σ implies its validity at any other $0 < \sigma < \infty$.

Appendix A. Proofs and Absolute Continuity

Proof of Proposition 2.1 The sup over a family of l.s.c., resp. convex, functions being l.s.c., resp. convex, it suffices to show that, for each $\phi \in \mathcal{C}_{c1}$, the reciprocal function $V_1^{-1}(\phi, \cdot)$ from (1.3), is weakly l.s.c. and convex. In this proof only, we pay a price for the simplifying convention $0/0 := 0$.

Let $F_n \rightarrow F$ weakly. Then $\int \phi^2 dF_n \rightarrow \int \phi^2 dF$. First assume $\int \phi^2 dF > 0$. Then $\int \phi^2 dF_n > 0$ eventually, and $V_1^{-1}(\phi, F_n) \rightarrow V_1^{-1}(\phi, F)$. Secondly suppose that $\int \phi^2 dF = 0$. If also $\int x \phi' dF = 0$, then $V_1^{-1}(\phi, F) = 0 \leq V_1^{-1}(\phi, F_n)$ for all n . If $\int x \phi' dF \neq 0$, then $\int \phi^2 dF_n \rightarrow 0$, $\int x \phi' dF_n \rightarrow \int x \phi' dF \neq 0$, hence $V_1^{-1}(\phi, F_n)$ tends to $\infty = V_1^{-1}(\phi, F)$.

Given F_1, F_2 , $s \in (0, 1)$, put $F = (1-s)F_1 + sF_2$. In case both $\int \phi^2 dF_j > 0$, we get $V_1^{-1}(\phi, F) \leq (1-s)V_1^{-1}(\phi, F_1) + sV_1^{-1}(\phi, F_2)$ from Huber (1981, Lemma 4.4). Secondly, let $\int \phi^2 dF_1 = 0 < \int \phi^2 dF_2$. Then, if $\int x \phi' dF_1 = 0$, hence $V_1^{-1}(\phi, F_1) = 0$, and $V_1^{-1}(\phi, F) = sV_1^{-1}(\phi, F_2) = (1-s)0 + sV_1^{-1}(\phi, F_2)$. If $\int x \phi' dF_1 \neq 0$, $V_1^{-1}(\phi, F_1) = \infty$ and $(1-s)\infty + sV_1^{-1}(\phi, F_2) \geq V_1^{-1}(\phi, F)$. Thirdly, let both $\int \phi^2 dF_j$ be zero. Then, if also both $\int x \phi' dF_j = 0$, we get $V_1^{-1}(\phi, F) = 0$. At least one $\int x \phi' dF_j$ nonzero implies that $(1-s)V_1^{-1}(\phi, F_1) + sV_1^{-1}(\phi, F_2) = \infty$. \square

Lemma A.1. For any finite measure F on \mathbb{B} , the class \mathcal{C}_{c1} is dense in $L_2(F)$. If $F(\{0\}) = 0$, the related class $\mathcal{D}_{c1} := \{x \mapsto x \phi'(x) \mid \phi \in \mathcal{C}_{c1}\}$ is dense in $L_2(F)$. There exist functions $0 \leq \phi_n \leq 1$ in \mathcal{C}_{c1} such that $\sup_{n,x} |x \phi_n'(x)| < \infty$, $\lim_n x \phi_n'(x) = 0$, and $\phi_n(x) \uparrow 1$, respectively $\phi_n(x) \downarrow 1_{\{x=0\}}$ pointwise.

Proof On the basis of Lusin's theorem, Rudin (1974, Thm. 3.14), it suffices to approximate the indicator of bounded intervals $(a, b]$.

For $\varepsilon \downarrow 0$ one may choose functions $g_\varepsilon \in \mathcal{C}_{c1}$ such that $0 \leq g_\varepsilon \leq 1$, $g_\varepsilon = 1$ on $[a + \varepsilon, b]$, $g_\varepsilon = 0$ on $(-\infty, a] \cup [b + \varepsilon, \infty)$. Then $g_\varepsilon \rightarrow 1_{(a,b]}$ pointwise, and $g_\varepsilon \rightarrow 1_{(a,b]}$ in $L_2(F)$ by dominated convergence.

Concerning denseness of \mathcal{D}_{c1} in $L_1(F_0)$, we may assume that $a > 0$. Drawing on the functions g_ε define $h_\varepsilon(x) := \int_{-\infty}^x y^{-1} g_\varepsilon(y) dy$. Then $h_\varepsilon \in \mathcal{C}_{c1}$ and, as before, $x h_\varepsilon' = g_\varepsilon \rightarrow 1_{(a,b]}$ in $L_2(F_0)$.

A possible choice of the functions ϕ_n , in the first case, is $\phi_n(x) = \phi(x/n)$, based on the function $2\phi(x) = 1 + \cos((|x| - \pi)_+ \wedge \pi)$, and, in the second case, $\phi_n(x) = \phi(nx)$, where $2\phi(x) = 1 + \cos(|x| \wedge \pi)$. \square

Absolute Continuity From real analysis, e.g., Rudin (1974, Ch.8), we recall: An \mathbb{R} -valued measure on the Borel σ -field \mathbb{B} of the real line is dominated by λ , the Lebesgue measure, iff its distribution function is absolutely continuous. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is absolutely continuous, if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any finite collection of disjoint segments $(a_i, b_i]$ of total length $\lambda(\bigcup (a_i, b_i]) < \delta$ it holds that $\sum_i |f(b_i) - f(a_i)| < \varepsilon$. Any absolutely continuous f has bounded variation on compact intervals $[a, b]$, the derivative f' exists a.e. λ , and $f(b) - f(a) = \int_a^b f' d\lambda$ where $\int_a^b |f'| d\lambda < \infty$. Integrability $f' \in L_1(\lambda)$, implying bounded variation on \mathbb{R} , and the limit $f(a) \rightarrow 0$ as $a \rightarrow -\infty$ require further conditions, respectively. These are obviously satisfied in the location case for absolutely continuous densities f such that $\mathcal{J}_1(F) < \infty$ for $dF = f d\lambda$, hence in particular $\int |f'| d\lambda < \infty$. If f and g are absolutely continuous, so is their product fg on any compact $[a, b]$. Thus, integration by parts holds: $f(b)g(b) - f(a)g(a) = \int_a^b f' g d\lambda + \int_a^b f g' d\lambda$ —a special case of Rieder (1994, Lemma C.2.1).

Proof of Theorem 2.2 First assume $\mathcal{J}_{s1}(F) < \infty$. On \mathcal{C}_{c1} define $T(\phi) := -\int x \phi' dF$, which operator is well defined, because $\int \phi^2 dF = 0$, in view of Definition 1.1, entails that $\int x \phi' dF = 0$.

Evaluated on \mathcal{C}_{c1} , T has operator norm $\sqrt{\mathcal{J}_{s1}(F)}$. \mathcal{C}_{c1} being dense in $L_2(F)$, T may be extended to $L_2(F)$ keeping its norm. By *Riesz–Fréchet* there exists some $g \in L_2(F)$, whose norm equals the operator norm of T , such that $T(\phi) = \int \phi g dF$ for all $\phi \in L_2(F)$, hence

$$-\int x \phi' dF = \int \phi g dF, \quad \phi \in \mathcal{C}_{c1}. \quad (\text{A.1})$$

Inserting ϕ_n from Lemma A.1, both choices, we obtain that, in addition to $\int g^2 dF = \mathcal{J}_{s1}(F)$,

$$\int g dF = 0, \quad g(0)F(\{0\}) = 0 \quad (\text{A.2})$$

In particular, the integrals in (A.1) and (A.2) may be restricted to $\mathbb{R} \setminus \{0\}$. Define the function

$$f(x) := \frac{1}{x} \int_{y \leq x} g(y) F_0(dy), \quad x \neq 0. \quad (\text{A.3})$$

Then, if $\phi_{-\infty}$ denotes the constant value of $\phi \in \mathcal{C}_{c1}$ left to the support of ϕ' , $\int \phi g dF = \int (\phi - \phi_{-\infty}) g dF_0$ and $\phi(x) - \phi_{-\infty} = \int_{0 \neq y \leq x} \phi'(y) \lambda(dy)$. Due to compact support of ϕ' , and $g \in L_2(F_0)$, the product $g(x) \phi'(y)$ is in $L_1(F_0(dx) \otimes \lambda(dy))$, and so $\int x \phi' dF_0 = -\int \int_{x > y \neq 0} g(x) \phi'(y) F_0(dx) \lambda(dy) = \int y f(y) \phi'(y) \lambda(dy)$ by *Fubini*; thus,

$$\int x \phi'(x) F_0(dx) = \int x \phi'(x) f(x) \lambda(dx), \quad \phi \in \mathcal{C}_{c1}. \quad (\text{A.4})$$

By denseness of \mathcal{D}_{c1} in $L_1(F_0)$, Lemma A.1, the LHS determines F_0 . As pointwise and dominated convergence $x h'_\varepsilon = g_\varepsilon \rightarrow 1_{(a,b]}$ has been established in that proof, also $f d\lambda$ on the RHS is completely determined by (A.4) if $f d\lambda$ is finite on any compact in $\mathbb{R} \setminus \{0\}$. But $\int_A^B |f| d\lambda \leq A^{-1} \int_A^B |x f(x)| \lambda(dx)$, which is bounded by $(B/A - 1) \int |g| dF_0 < \infty$ for $A > 0$, and likewise for $B < 0$. Thus we conclude from (A.4) that

$$dF_0 = f d\lambda. \quad (\text{A.5})$$

Since F_0 is nonnegative, in fact $f \geq 0$ a.e. λ . Absolute continuity of the function m ,

$$m(x) := \int_{y \leq x} g(y) F_0(dy) = \int_{y \leq x} g(y) f(y) \lambda(dy). \quad (\text{A.6})$$

follows from $\int |g| f d\lambda = \int |g| dF_0 < \infty$. As $m(x) = x f(x)$ for $x \neq 0$, differentiability of f a.e. λ (for $x \neq 0$) is entailed by that of m , and

$$g(x) = 1 + x f'(x) / f(x) \quad \text{a.e. } F_0(dx). \quad (\text{A.7})$$

This completes the identification of g under F , and i)–iii) are proved.

Conversely, assume i)–iii). By ii), $m(x) = x f(x)$ is absolutely continuous. Differentiability of m at $x \neq 0$ implies that of f , and $m' = f + x f'$. For λ -densities, necessarily $\lambda(f = 0, f' \neq 0) = 0$, hence also $\lambda(f =$

$0, m' \neq 0) = 0$. With $-\Lambda = m'/f = 1 + x f'/f$ a.e. F_0 , we have $\int |m'| d\lambda = \int |\Lambda| dF_0 < \infty$ by iii). Thus, m and its measure $m' d\lambda = -\Lambda dF_0$ are of bounded variation on \mathbb{R} .

By Hölder inequality, $|m(y) - m(x)|^2 \leq |F(y) - F(x)| \int \Lambda^2 dF_0$, so $m(x)$ for $x \rightarrow \infty$ is a Cauchy sequence. But $\lim_{x \rightarrow \infty} m(x)$ must be zero since otherwise $f(x) \sim 1/x$ for $x \rightarrow \infty$ would not integrate. The same holding for $x \rightarrow -\infty$, we obtain

$$\int m' d\lambda = 0. \quad (\text{A.8})$$

For $\phi \in \mathcal{C}_{c1}$, the function $\phi - \phi_{-\infty}$ and corresponding measure $\phi' d\lambda$ have bounded variation on \mathbb{R} . Thus integration by parts in the general form of Rieder (1994, Lem. C.2.1) yields $\int \phi' m d\lambda = - \int \phi m' d\lambda$, such that

$$\int x \phi' dF = \int \phi' m d\lambda = - \int \phi m' d\lambda = \int \phi \Lambda dF_0. \quad (\text{A.9})$$

Applying Cauchy-Schwarz, we get

$$\left(\int x \phi' dF \right)^2 = \left(\int \phi \Lambda dF_0 \right)^2 \leq \int \phi^2 dF_0 \int \Lambda^2 dF_0, \quad (\text{A.10})$$

where $\int \Lambda^2 dF_0$ is finite by iii). It follows that $\mathcal{I}_{s1}(F) < \infty$. \square

Proof of Proposition 3.1 We decompose $\|\sqrt{dF_{\sigma+t}} - \sqrt{dF_{\sigma}}(1 + \frac{1}{2}t\Lambda_{\sigma})\|$ into the following sum,

$$\left\| \left(\sqrt{dF_{\sigma+t}} - \sqrt{dF_{\sigma}}(1 + \frac{1}{2}t\Lambda_{\sigma}) \right) 1_{\{0\}^c} \right\| + \left\| \left(\sqrt{dF_{\sigma+t}} - \sqrt{dF_{\sigma}}(1 + \frac{1}{2}t\Lambda_{\sigma}) \right) 1_{\{0\}} \right\|, \quad (\text{A.11})$$

The first summand is $o(t)$ by Swensen (1980). The second is 0, since $F_{\sigma}(\{0\}) = F(\{0\})$ and $\Lambda_{\sigma}(0) = 0$. \square

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